Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Computational and Applied Mathematics 208 (2007) 391–403
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

A new two-level implicit discretization of $O(k^2 + kh^2 + h^4)$ for the solution of singularly perturbed two-space dimensional non-linear parabolic equations

R.K. Mohanty*, Swarn Singh[1]

Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110 007, India

Received 10 January 2006; received in revised form 7 September 2006

Abstract

We propose a new two-level implicit difference method of $O(k^2 + kh^2 + h^4)$ for the solution of singularly perturbed non-linear parabolic differential equation $\varepsilon(u_{xx}+u_{yy})=f(x, y, t, u, u_x, u_y, u_t), 0<x, y<1, t>0$ subject to appropriate initial and Dirichlet boundary conditions, where $k>0$ and $h>0$ are grid sizes in time and space directions, respectively, and $\varepsilon>0$ is a small parameter. We also develop new methods of $O(kh^2 + h^4)$ for the estimates of $(\partial u/\partial x)$ and $(\partial u/\partial y)$. In all cases, we use 9-spatial grid points and a single computational cell. The proposed methods are directly applicable to singular problems. We do not require any special scheme to solve singular problems. We also discuss alternating direction implicit (ADI) method for solving diffusion equation in polar cylindrical coordinates. This method permits multiple use of the one-dimensional tri-diagonal algorithm with a considerable saving in computing time, and produces a very efficient solver. It is shown that the ADI method is unconditionally stable. Numerical experiments are conducted to test the high accuracy of the proposed methods and compared with the exact solutions.

MSC: 65M06; G.1.8

Keywords: Implicit scheme; High order compact scheme; Singularly perturbed; Non-linear parabolic equation; Normal derivatives; Heat equation; Polar cylindrical coordinates; Burgers' equation

1. Introduction

Many numerical methods have been suggested for the solution of the two-space dimensional parabolic equations. Two types of finite difference schemes which have been studied previously are explicit difference schemes and implicit difference schemes. In analyzing time-dependent parabolic problems using numerical methods, the authors found that the machine time required was large and cost prohibitive. This was true regardless of the method used. Explicit methods are conditionally stable and when were used, the time step was restricted to a value very much smaller than the maximum allowable for the solution of the parabolic equation. On the other hand, when implicit methods were tried, the problem of solving large scale systems was encountered. However, in most of the cases these implicit methods are unconditionally stable.

* Corresponding author. Tel.: +91 11 2808 0912.
E-mail addresses: rmohanty@maths.du.ac.in, mohantyranjan@hotmail.com (R.K. Mohanty), sswarn2005@yahoo.co.in (S. Singh).
[1] Permanent address: Department of Mathematics, Sri Venkateswara College, University of Delhi, New Delhi 110 021, India.

0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.10.023

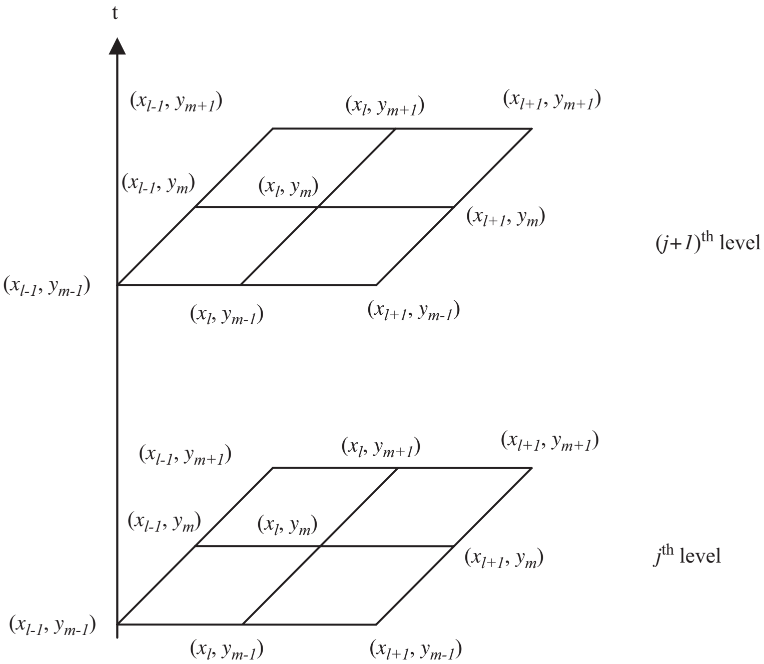

Fig. 1. The 9-spatial grid points.

The problem which will be considered here is the numerical solution of singularly perturbed non-linear parabolic equation of the form

$$\varepsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f(x,y,t,u,u_x,u_y,u_t),\quad 0<x,\quad y<1,\quad t>0 \tag{1}$$

with the initial and boundary conditions as

$$u(x,y,0)=u_0(x,y),\quad 0\leqslant x,\quad y\leqslant 1, \tag{2}$$

$$u(0,y,t)=g_0(y,t),\quad u(1,y,t)=g_1(y,t),\quad 0\leqslant y\leqslant 1,\quad t\geqslant 0, \tag{3a}$$

$$u(x,0,t)=h_0(x,t),\quad u(x,1,t)=h_1(x,t),\quad 0\leqslant x\leqslant 1,\quad t\geqslant 0, \tag{3b}$$

where $\varepsilon>0$ is a small parameter and u_0, g_0, g_1, h_0, h_1 are given functions of sufficient smoothness.

Two-space dimensional singularly perturbed parabolic equations with first derivative terms are encountered in heat transfer, neutron diffusion and fluid flow problem (see [14]). Both explicit and implicit difference methods have been developed for the differential equation (1). The explicit schemes are usually very time-consuming due to the stability restriction. The implicit schemes are unconditionally stable, thus allowing a large time step. The disadvantage however, is that the solution values for all grid points must be obtained simultaneously. There is considerable interest in developing high order finite difference schemes for the linear unsteady diffusion–convection equation (see [13,12,4,15,17,5,6]). The alternating direction implicit (ADI) methods have been proposed to solve linear parabolic equations (see [7,2]). In 1996, Mohanty and Jain [8] have proposed a two-level implicit high accuracy difference method for the solution of two-space dimensional non-linear parabolic equations. Later, Mohanty et al. [9], have derived a method for the estimates of $(\partial u/\partial n)$. Both methods require 9-spatial grid points and a single computational cell. However, their methods are not directly applicable to singular parabolic problems. A special technique is required to solve parabolic problem in polar coordinates. In this paper, using 9-spatial grid points and a single computational cell (see Fig. 1), we derive new formulas of order 2 in time and 4 in space coordinates for the solution of non-linear parabolic equation (1) and the estimates of $(\partial u/\partial x)$ and $(\partial u/\partial y)$.

The proposed methods are directly applicable to parabolic equations in polar coordinates. We do not need more than 9-spatial grid points to discretize the differential equation (1). Recently, Mohanty and Singh [10,11] have proposed new high accuracy arithmetic average discretizations for singularly perturbed 1-D parabolic and 2-D elliptic non-linear partial differential equations. In next two sections, we give mathematical details of the methods. In Section 4, we discuss ADI method for the solution of heat equation in cylindrical polar coordinates. The proposed ADI method is shown through a discrete Fourier analysis to be unconditionally stable. In Section 5, numerical experiments are performed to test the accuracy and efficiency of the proposed numerical methods and compared with the difference method of order 2 in time and order 2 in space. Final remarks are given in Section 6.

2. Two-level implicit scheme

Assume that the solution domain is covered by a rectangular grid with spacing $h>0$ and $k>0$ in space and time coordinates, respectively. We replace the solution region by a set of grid points (x_l, y_m, t_j) where $x_l = lh$, $y_m = mh$, $t_j = jk$, with $l, m = 0, 1, 2, \ldots, N+1$, $(N+1)h = 1$, N is a positive integer and $j = 0, 1, 2, \ldots$. The mesh ratio parameter is given by $\lambda = (k/h^2)$. Let $u_{l,m}^j$ and $U_{l,m}^j$ be the approximate and exact solution values of $u(x, y, t)$ at the grid point (x_l, y_m, t_j), respectively. Here we denote $t_j = jk$ as jth-level (or, first time level) and $t_{j+1} = t_j + k$ as $(j+1)$th-level (or, second time level) (see Fig. 1).

We require the following approximations:

$$\bar{t}_j = t_j + \theta k, \quad 0 \leqslant \theta \leqslant 1, \tag{4}$$

$$\overline{U}_{l,m}^j = \theta U_{l,m}^{j+1} + (1-\theta) U_{l,m}^j, \tag{5a}$$

$$\overline{U}_{l\pm 1,m}^j = \theta U_{l\pm 1,m}^{j+1} + (1-\theta) U_{l\pm 1,m}^j, \tag{5b}$$

$$\overline{U}_{l,m\pm 1}^j = \theta U_{l,m\pm 1}^{j+1} + (1-\theta) U_{l,m\pm 1}^j, \tag{5c}$$

$$\overline{U}_{l\pm 1/2,m}^j = (\overline{U}_{l\pm 1,m}^j + \overline{U}_{l,m}^j)/2, \tag{5d}$$

$$\overline{U}_{l,m\pm 1/2}^j = (\overline{U}_{l,m\pm 1}^j + \overline{U}_{l,m}^j)/2, \tag{5e}$$

$$\overline{U}_{x_{l,m}}^j = (\overline{U}_{l+1,m}^j - \overline{U}_{l-1,m}^j)/(2h), \tag{6a}$$

$$\overline{U}_{x_{l\pm 1/2,m}}^j = \pm(\overline{U}_{l\pm 1,m}^j - \overline{U}_{l,m}^j)/h, \tag{6b}$$

$$\overline{U}_{x_{l,m\pm 1/2}}^j = (\overline{U}_{l+1,m\pm 1}^j - \overline{U}_{l-1,m\pm 1}^j + \overline{U}_{l+1,m}^j - \overline{U}_{l-1,m}^j)/(4h), \tag{6c}$$

$$\overline{U}_{y_{l,m}}^j = (\overline{U}_{l,m+1}^j - \overline{U}_{l,m-1}^j)/(2h), \tag{7a}$$

$$\overline{U}_{y_{l\pm 1/2,m}}^j = (\overline{U}_{l\pm 1,m+1}^j - \overline{U}_{l\pm 1,m-1}^j + \overline{U}_{l,m+1}^j - \overline{U}_{l,m-1}^j)/(4h), \tag{7b}$$

$$\overline{U}_{y_{l,m\pm 1/2}}^j = \pm(\overline{U}_{l,m\pm 1}^j - \overline{U}_{l,m}^j)/h, \tag{7c}$$

$$\overline{U}_{t_{l,m}}^j = (U_{l,m}^{j+1} - U_{l,m}^j)/k, \tag{8a}$$

$$\overline{U}_{t_{l\pm 1,m}}^j = (U_{l\pm 1,m}^{j+1} - U_{l\pm 1,m}^j)/k, \tag{8b}$$

$$\overline{U}^{j}_{t_{l,m\pm1}} = (U^{j+1}_{l,m\pm1} - U^{j}_{l,m\pm1})/k, \tag{8c}$$

$$\overline{U}^{j}_{t_{l\pm1/2,m}} = (U^{j+1}_{l\pm1,m} + U^{j+1}_{l,m} - U^{j}_{l\pm1,m} - U^{j}_{l,m})/(2k), \tag{8d}$$

$$\overline{U}^{j}_{t_{l,m\pm1/2}} = (U^{j+1}_{l,m\pm1} + U^{j+1}_{l,m} - U^{j}_{l,m\pm1} - U^{j}_{l,m})/(2k). \tag{8e}$$

Next we define

$$\overline{F}^{j}_{l+1/2,m} = f(x_{l+1/2}, y_m, \bar{t}_j, \overline{U}^{j}_{l+1/2,m}, \overline{U}^{j}_{x_{l+1/2,m}}, \overline{U}^{j}_{y_{l+1/2,m}}, \overline{U}^{j}_{t_{l+1/2,m}}), \tag{9a}$$

$$\overline{F}^{j}_{l-1/2,m} = f(x_{l-1/2}, y_m, \bar{t}_j, \overline{U}^{j}_{l-1/2,m}, \overline{U}^{j}_{x_{l-1/2,m}}, \overline{U}^{j}_{y_{l-1/2,m}}, \overline{U}^{j}_{t_{l-1/2,m}}), \tag{9b}$$

$$\overline{F}^{j}_{l,m+1/2} = f(x_l, y_{m+1/2}, \bar{t}_j, \overline{U}^{j}_{l,m+1/2}, \overline{U}^{j}_{x_{l,m+1/2}}, \overline{U}^{j}_{y_{l,m+1/2}}, \overline{U}^{j}_{t_{l,m+1/2}}), \tag{9c}$$

$$\overline{F}^{j}_{l,m-1/2} = f(x_l, y_{m-1/2}, \bar{t}_j, \overline{U}^{j}_{l,m-1/2}, \overline{U}^{j}_{x_{l,m-1/2}}, \overline{U}^{j}_{y_{l,m-1/2}}, \overline{U}^{j}_{t_{l,m-1/2}}), \tag{9d}$$

$$\overline{\overline{U}}^{j}_{l,m} = \overline{U}^{j}_{l,m} + a_1 h^2 (\overline{F}^{j}_{l+1/2,m} + \overline{F}^{j}_{l-1/2,m} + \overline{F}^{j}_{l,m+1/2} + \overline{F}^{j}_{l,m-1/2}), \tag{10a}$$

$$\overline{\overline{U}}^{j}_{x_{l,m}} = \overline{U}^{j}_{x_{l,m}} + a_2 h (\overline{F}^{j}_{l+1/2,m} - \overline{F}^{j}_{l-1/2,m}), \tag{10b}$$

$$\overline{\overline{U}}^{j}_{y_{l,m}} = \overline{U}^{j}_{y_{l,m}} + a_3 h (\overline{F}^{j}_{l,m+1/2} - \overline{F}^{j}_{l,m-1/2}), \tag{10c}$$

$$\overline{\overline{U}}^{j}_{t_{l,m}} = \overline{U}^{j}_{t_{l,m}} + a_4 (\overline{U}^{j}_{t_{l+1,m}} + \overline{U}^{j}_{t_{l-1,m}} + \overline{U}^{j}_{t_{l,m+1}} + \overline{U}^{j}_{t_{l,m-1}} - 4\overline{U}^{j}_{t_{l,m}}), \tag{10d}$$

where

$$\theta = \frac{1}{2}, \quad a_1 = \frac{1}{16\varepsilon}, \quad a_2 = \frac{1}{4\varepsilon}, \quad a_3 = \frac{1}{4\varepsilon}, \quad a_4 = \frac{1}{4}.$$

Finally, we define

$$\overline{\overline{F}}^{j}_{l,m} = f\left(x_l, y_m, \bar{t}_j, \overline{\overline{U}}^{j}_{l,m}, \overline{\overline{U}}^{j}_{x_{l,m}}, \overline{\overline{U}}^{j}_{y_{l,m}}, \overline{\overline{U}}^{j}_{t_{l,m}}\right). \tag{11}$$

Then at each internal grid point (x_l, y_m, t_j), the proposed parabolic differential equation (1) is discretized by

$$\varepsilon\left[\delta_x^2 + \delta_y^2 + \frac{1}{6}\delta_x^2\delta_y^2\right]\overline{U}^{j}_{l,m} = \frac{h^2}{3}\left[\overline{F}^{j}_{l+1/2,m} + \overline{F}^{j}_{l-1/2,m} + \overline{F}^{j}_{l,m+1/2} + \overline{F}^{j}_{l,m-1/2} - \overline{\overline{F}}^{j}_{l,m}\right] + \overline{T}^{j}_{l,m}, \tag{12}$$

where $\delta_x U_l = (U_{l+1/2} - U_{l-1/2})$ is the central difference operator with respect to x-direction, etc. and $\overline{T}^{j}_{l,m} = \mathrm{O}(k^2h^2 + kh^4 + h^6)$.

Next for the estimates of $(\partial u/\partial x)$ and $(\partial u/\partial y)$, we need the following approximations. Let

$$\hat{U}^{j+1}_{l\pm1/2,m} = (U^{j+1}_{l\pm1,m} + U^{j+1}_{l,m})/2, \tag{13a}$$

$$\hat{U}^{j+1}_{l,m\pm1/2} = (U^{j+1}_{l,m\pm1} + U^{j+1}_{l,m})/2, \tag{13b}$$

$$\hat{U}^{j+1}_{x_{l\pm1/2,m}} = \pm(U^{j+1}_{l\pm1,m} - U^{j+1}_{l,m})/h, \tag{14a}$$

$$\hat{U}^{j+1}_{x_{l,m\pm1/2}} = (U^{j+1}_{l+1,m\pm1} - U^{j+1}_{l-1,m\pm1} + U^{j+1}_{l+1,m} - U^{j+1}_{l-1,m})/(4h), \tag{14b}$$

$$\hat{U}^{j+1}_{y_{l\pm 1/2,m}} = (U^{j+1}_{l\pm 1,m+1} - U^{j+1}_{l\pm 1,m-1} + U^{j+1}_{l,m+1} - U^{j+1}_{l,m-1})/(4h), \tag{15a}$$

$$\hat{U}^{j+1}_{y_{l,m\pm 1/2}} = \pm(U^{j+1}_{l,m\pm 1} - U^{j+1}_{l,m})/h, \tag{15b}$$

$$\hat{U}^{j+1}_{t_{l\pm 1/2,m}} = (U^{j+1}_{l\pm 1,m} + U^{j+1}_{l,m} - U^{j}_{l\pm 1,m} - U^{j}_{l,m})/(2k), \tag{16a}$$

$$\hat{U}^{j+1}_{t_{l,m\pm 1/2}} = (U^{j+1}_{l,m\pm 1} + U^{j+1}_{l,m} - U^{j}_{l,m\pm 1} - U^{j}_{l,m})/(2k). \tag{16b}$$

Then we define

$$\hat{F}^{j+1}_{l+1/2,m} = f(x_{l+1/2}, y_m, t_{j+1}, \hat{U}^{j+1}_{l+1/2,m}, \hat{U}^{j+1}_{x_{l+1/2,m}}, \hat{U}^{j+1}_{y_{l+1/2,m}}, \hat{U}^{j+1}_{t_{l+1/2,m}}), \tag{17a}$$

$$\hat{F}^{j+1}_{l-1/2,m} = f(x_{l-1/2}, y_m, t_{j+1}, \hat{U}^{j+1}_{l-1/2,m}, \hat{U}^{j+1}_{x_{l-1/2,m}}, \hat{U}^{j+1}_{y_{l-1/2,m}}, \hat{U}^{j+1}_{t_{l-1/2,m}}), \tag{17b}$$

$$\hat{F}^{j+1}_{l,m+1/2} = f(x_l, y_{m+1/2}, t_{j+1}, \hat{U}^{j+1}_{l,m+1/2}, \hat{U}^{j+1}_{x_{l,m+1/2}}, \hat{U}^{j+1}_{y_{l,m+1/2}}, \hat{U}^{j+1}_{t_{l,m+1/2}}), \tag{17c}$$

$$\hat{F}^{j+1}_{l,m-1/2} = f(x_l, y_{m-1/2}, t_{j+1}, \hat{U}^{j+1}_{l,m-1/2}, \hat{U}^{j+1}_{x_{l,m-1/2}}, \hat{U}^{j+1}_{y_{l,m-1/2}}, \hat{U}^{j+1}_{t_{l,m-1/2}}). \tag{17d}$$

Following the techniques given by Stephenson [16], the methods for the estimates of $(\partial u/\partial x)$ and $(\partial u/\partial y)$ for the differential equation (1) are given by

$$\begin{aligned} U^{j+1}_{x_{l,m}} = &\frac{1}{12h}\left[U^{j+1}_{l+1,m+1} + U^{j+1}_{l+1,m-1} - U^{j+1}_{l-1,m+1} - U^{j+1}_{l-1,m-1} + 4(U^{j+1}_{l+1,m} - U^{j+1}_{l-1,m})\right] \\ &- \frac{h}{6\varepsilon}\left[\hat{F}^{j+1}_{l+1/2,m} - \hat{F}^{j+1}_{l-1/2,m}\right] + \hat{T}^{j+1}_{x_{l,m}}, \end{aligned} \tag{18a}$$

$$\begin{aligned} U^{j+1}_{y_{l,m}} = &\frac{1}{12h}\left[U^{j+1}_{l+1,m+1} - U^{j+1}_{l+1,m-1} + U^{j+1}_{l-1,m+1} - U^{j+1}_{l-1,m-1} + 4(U^{j+1}_{l,m+1} - U^{j+1}_{l,m-1})\right] \\ &- \frac{h}{6\varepsilon}\left[\hat{F}^{j+1}_{l,m+1/2} - \hat{F}^{j+1}_{l,m-1/2}\right] + \hat{T}^{j+1}_{y_{l,m}}, \end{aligned} \tag{18b}$$

where $\hat{T}^{j+1}_{x_{l,m}} = \mathrm{O}(kh^2 + h^4)$ and $\hat{T}^{j+1}_{y_{l,m}} = \mathrm{O}(kh^2 + h^4)$.

3. Derivation of the method

For the derivation of the new method, we simply follow the techniques given by Mohanty and Singh [11] and Chawla and Shivakumar [1].

At the grid point (x_l, y_m, t_j), we denote

$$U_{abc} = \frac{\partial^{a+b+c} U}{(\partial x)^a (\partial y)^b (\partial t)^c}, \tag{19}$$

$$G = \frac{\partial f}{\partial t}, \quad H = \frac{\partial f}{\partial u}, \quad I = \frac{\partial f}{\partial u_x}, \quad J = \frac{\partial f}{\partial u_y}, \quad K = \frac{\partial f}{\partial u_t}. \tag{20}$$

At the grid point (x_l, y_m, t_j), we denote

$$\varepsilon\left(\frac{\partial^2 U^j_{l,m}}{\partial x^2} + \frac{\partial^2 U^j_{l,m}}{\partial y^2}\right) = f\left(x_l, y_m, t_j, U^j_{l,m}, U^j_{x_{l,m}}, U^j_{y_{l,m}}, U^j_{t_{l,m}}\right) \equiv F^j_{l,m}. \tag{21}$$

By the help of the Taylor expansion, we obtain

$$\varepsilon\left[\delta_x^2+\delta_y^2+\frac{1}{6}\delta_x^2\delta_y^2\right]U_{l,m}^j=\frac{h^2}{3}\left[F_{l+1/2,m}^j+F_{l-1/2,m}^j+F_{l,m+1/2}^j+F_{l,m-1/2}^j-F_{l,m}^j\right]+\mathrm{O}(h^6). \tag{22}$$

Now differentiating the differential equation (1) with respect to 't' at the grid point (x_l, y_m, t_j), we obtain a relation of the form

$$-KU_{002}=G+HU_{001}+IU_{101}+JU_{011}-\varepsilon(U_{201}+U_{021}). \tag{23}$$

By the help of the approximations (4)–(8e) and simplifying (9a), we get

$$\begin{aligned}\overline{F}_{l+1/2,m}^j=f\Bigg(&x_{l+1/2}, y_m, t_j+\theta k, U_{l+1/2,m}^j+\theta kU_{t_{l,m}}^j+\frac{h^2}{8}U_{xx_{l,m}}^j+\mathrm{O}(hk+h^3),\\ &U_{x_{l+1/2,m}}^j+\theta kU_{xt_{l,m}}^j+\frac{h^2}{24}U_{xxx_{l,m}}^j+\mathrm{O}(hk+h^3),\\ &U_{y_{l+1/2,m}}^j+\theta kU_{yt_{l,m}}^j+\frac{h^2}{24}(3U_{xxy_{l,m}}^j+4U_{yyy_{l,m}}^j)+\mathrm{O}(hk+h^3),\\ &U_{t_{l+1/2,m}}^j+\frac{k}{2}U_{tt_{l,m}}^j+\frac{h^2}{8}U_{xxt_{l,m}}^j+\mathrm{O}(hk+h^3)\Bigg)\\ =F_{l+1/2,m}^j&+\theta kG_{l+1/2,m}^j+\left(\theta kU_{t_{l,m}}^j+\frac{h^2}{8}U_{xx_{l,m}}^j+\mathrm{O}(hk+h^3)\right)H_{l+1/2,m}^j\\ &+\left(\theta kU_{xt_{l,m}}^j+\frac{h^2}{24}U_{xxx_{l,m}}^j+\mathrm{O}(hk+h^3)\right)I_{l+1/2,m}^j\\ &+\left(\theta kU_{yt_{l,m}}^j+\frac{h^2}{24}(3U_{xxy_{l,m}}^j+4U_{yyy_{l,m}}^j)+\mathrm{O}(hk+h^3)\right)J_{l+1/2,m}^j\\ &+\left(\frac{k}{2}U_{tt_{l,m}}^j+\frac{h^2}{8}U_{xxt_{l,m}}^j+\mathrm{O}(hk+h^3)\right)K_{l+1/2,m}^j.\end{aligned}$$

Now using the approximations

$$G_{l+1/2,m}^j=G_{l,m}^j+\frac{h}{2}G_{x_{l,m}}^j+\mathrm{O}(h^2),\quad H_{l+1/2,m}^j=H_{l,m}^j+\frac{h}{2}H_{x_{l,m}}^j+\mathrm{O}(h^2)\quad \text{etc.},$$

we get

$$\overline{F}_{l+1/2,m}^j=F_{l+1/2,m}^j+\frac{k}{2}T_1+\frac{h^2}{24}T_2+\mathrm{O}(hk+h^3). \tag{24a}$$

Similarly, simplifying (9b)–(9d), we obtain

$$\overline{F}_{l-1/2,m}^j=F_{l-1/2,m}^j+\frac{k}{2}T_1+\frac{h^2}{24}T_2-\mathrm{O}(hk+h^3), \tag{24b}$$

$$\overline{F}_{l,m+1/2}^j=F_{l,m+1/2}^j+\frac{k}{2}T_1+\frac{h^2}{24}T_3+\mathrm{O}(hk+h^3), \tag{24c}$$

$$\overline{F}_{l,m-1/2}^j=F_{l,m-1/2}^j+\frac{k}{2}T_1+\frac{h^2}{24}T_3-\mathrm{O}(hk+h^3), \tag{24d}$$

where

$$\begin{aligned}T_1&=2\theta(G+U_{001}H+U_{101}I+U_{011}J)+U_{002}K,\\ T_2&=3U_{200}H+U_{300}I+(3U_{210}+4U_{030})J+3U_{201}K,\\ T_3&=3U_{020}H+(4U_{300}+3U_{120})I+U_{030}J+3U_{021}K.\end{aligned}$$

With the help of (24a)–(24d), from (10a)–(10d), we obtain

$$\overline{\overline{U}}^{j}_{l,m} = U^{j}_{l,m} + \theta k U_{001} + 4\varepsilon a_1 h^2 (U_{200} + U_{020}) + \mathrm{O}(k^2 + kh^2 + h^4), \tag{25a}$$

$$\overline{\overline{U}}^{j}_{x_{l,m}} = U^{j}_{x_{l,m}} + \theta k U_{101} + \frac{h^2}{6}[(1 + 6\varepsilon a_2) U_{300} + 6\varepsilon a_2 U_{120}] + \mathrm{O}(kh^2 + h^4), \tag{25b}$$

$$\overline{\overline{U}}^{j}_{y_{l,m}} = U^{j}_{y_{l,m}} + \theta k U_{011} + \frac{h^2}{6}[(1 + 6\varepsilon a_3) U_{030} + 6\varepsilon a_3 U_{210}] + \mathrm{O}(kh^2 + h^4), \tag{25c}$$

$$\overline{\overline{U}}^{j}_{t_{l,m}} = U^{j}_{t_{l,m}} + \frac{k}{2} U_{002} + a_4 h^2 (U_{201} + U_{021}) + \mathrm{O}(k^2 + h^4). \tag{25d}$$

Then, simplifying (11), we obtain

$$\overline{\overline{F}}^{j}_{l,m} = F^{j}_{l,m} + \frac{k}{2} T_1 + \frac{h^2}{6} T_4 + \mathrm{O}(k^2 + kh^2 + h^4), \tag{26}$$

where

$$\begin{aligned} T_4 = {} & 24\varepsilon a_1 (U_{200} + U_{020}) H + [(1 + 6\varepsilon a_2) U_{300} + 6\varepsilon a_2 U_{120}] I \\ & + [(1 + 6\varepsilon a_3) U_{030} + 6\varepsilon a_3 U_{210}] J + 6 a_4 (U_{201} + U_{021}) K. \end{aligned}$$

Further, we may re-write

$$\varepsilon(\delta_x^2 + \delta_y^2 + \tfrac{1}{6}\delta_x^2\delta_y^2)\overline{U}^{j}_{l,m} = \varepsilon(\delta_x^2 + \delta_y^2 + \tfrac{1}{6}\delta_x^2\delta_y^2) U^{j}_{l,m} + \varepsilon\theta k h^2 (U_{201} + U_{021}) + \mathrm{O}(k^2h^2 + kh^4 + h^6). \tag{27}$$

Finally, by the help of relations (23), (24a)–(24d), (26) and (27), from (12) and (22), we obtain the local truncation error as

$$\overline{T}^{j}_{l,m} = -kh^2 \left(\frac{1}{2} - \theta\right) U_{002} K - \frac{h^4}{36}(T_2 + T_3 - 2T_4) + \mathrm{O}(k^2h^2 + kh^4 + h^6). \tag{28}$$

The proposed difference method (12) to be of $\mathrm{O}(k^2 + kh^2 + h^4)$, the coefficients of kh^2 and h^4 in (28) must be zero, hence

$$\tfrac{1}{2} - \theta = 0 \tag{29a}$$

and $T_2 + T_3 - 2T_4 = 0$

or

$$\begin{aligned} & (1 - 16\varepsilon a_1)(U_{200} + U_{020}) H + (1 - 4\varepsilon a_2)(U_{300} + U_{120}) I + (1 - 4\varepsilon a_3)(U_{030} + U_{210}) J \\ & \quad + (1 - 4 a_4)(U_{201} + U_{021}) K = 0. \end{aligned} \tag{29b}$$

Thus we obtain the values of parameters $\theta = \frac{1}{2}$, $a_1 = 1/16\varepsilon$, $a_2 = 1/4\varepsilon$, $a_3 = 1/4\varepsilon$, $a_4 = 1/4$ for which the proposed method (12) becomes $\mathrm{O}(k^2 + kh^2 + h^4)$ and $\overline{T}^{j}_{l,m} = \mathrm{O}(k^2h^2 + kh^4 + h^6)$.

Next we discuss the methods of $\mathrm{O}(kh^2 + h^4)$ for the estimates of $(\partial u/\partial x)$ and $(\partial u/\partial y)$. Once the solution u has been obtained at $(j+1)$th level, one may compute these values using the central difference approximations

$$u^{j+1}_{x_{l,m}} = (u^{j+1}_{l+1,m} - u^{j+1}_{l-1,m})/(2h), \tag{30a}$$

$$u^{j+1}_{y_{l,m}} = (u^{j+1}_{l,m+1} - u^{j+1}_{l,m-1})/(2h). \tag{30b}$$

It has been verified that the standard central difference approximations yield $\mathrm{O}(h^2)$ accurate results irrespective of whether difference method (12), which is of $\mathrm{O}(k^2 + kh^2 + h^4)$ or difference method of $\mathrm{O}(k^2 + h^2)$ is used to solve the parabolic equation (1). New difference formulas of $\mathrm{O}(kh^2 + h^4)$ for computing the numerical values of u_x and u_y are proposed. These new formulas are found to yield $\mathrm{O}(h^4)$-accuracy for a fixed mesh ratio parameter λ, when used in conjunction with two-level implicit method (12).

By the help of Taylor series expansion, we obtain

$$U_{x_{l,m}}^{j+1} = \frac{1}{12h}\left[U_{l+1,m+1}^{j+1} + U_{l+1,m-1}^{j+1} - U_{l-1,m+1}^{j+1} - U_{l-1,m-1}^{j+1} + 4(U_{l+1,m}^{j+1} - U_{l-1,m}^{j+1})\right] - \frac{h}{6\varepsilon}\left[F_{l+1/2,m}^{j+1} - F_{l-1/2,m}^{j+1}\right] + \mathrm{O}(h^4), \tag{31a}$$

$$U_{y_{l,m}}^{j+1} = \frac{1}{12h}\left[U_{l+1,m+1}^{j+1} - U_{l+1,m-1}^{j+1} + U_{l-1,m+1}^{j+1} - U_{l-1,m-1}^{j+1} + 4(U_{l,m+1}^{j+1} - U_{l,m-1}^{j+1})\right] - \frac{h}{6\varepsilon}\left[F_{l,m+1/2}^{j+1} - F_{l,m-1/2}^{j+1}\right] + \mathrm{O}(h^4), \tag{31b}$$

where

$$F_{l\pm1/2,m}^{j+1} = f\left(x_{l\pm1/2}, y_m, t_{j+1}, U_{l\pm1/2,m}^{j+1}, U_{x_{l\pm1/2,m}}^{j+1}, U_{y_{l\pm1/2,m}}^{j+1}, U_{t_{l\pm1/2,m}}^{j+1}\right),$$

$$F_{l,m\pm1/2}^{j+1} = f\left(x_l, y_{m\pm1/2}, t_{j+1}, U_{l,m\pm1/2}^{j+1}, U_{x_{l,m\pm1/2}}^{j+1}, U_{y_{l,m\pm1/2}}^{j+1}, U_{t_{l,m\pm1/2}}^{j+1}\right).$$

By the help of the approximations (13a)–(16b), from (17a)–(17d), we obtain

$$\hat{F}_{l\pm1/2,m}^{j+1} = F_{l\pm1/2,m}^{j+1} + \mathrm{O}(k+h^2), \tag{32a}$$

$$\hat{F}_{l,m\pm1/2}^{j+1} = F_{l,m\pm1/2}^{j+1} + \mathrm{O}(k+h^2). \tag{32b}$$

With the help of the approximations (32a), (32b) and using relations (31a), (31b), from (18a) and (18b), it is easy to verify that $\hat{T}_{x_{l,m}}^{j+1} = \mathrm{O}(kh^2+h^4)$ and $\hat{T}_{y_{l,m}}^{j+1} = \mathrm{O}(kh^2+h^4)$.

Note that the matrices represented by the new formulas (12) and (18a), (18b) are tri-block diagonal and diagonal, respectively. The formulas are of $\mathrm{O}(k^2+kh^2+h^4)$ accuracy and free from the terms $(1/x_{l\pm1})$ and $(1/y_{m\pm1})$, hence very easily solved for $l, m = 1(1)N$ in the region $0 < x,\ y < 1,\ t > 0$. If the differential equation is linear, we can solve the linear system by using ADI method, whereas for non-linear case, we can use Newton–Raphson method. The proposed numerical methods are directly applicable to singular parabolic problems in the region $0 < x,\ y < 1,\ t > 0$. It is mentioned here that in order to get $\mathrm{O}(kh^2+h^4)$ numerical solution of $(\partial u/\partial x)$ and $(\partial u/\partial y)$ from (18a) and (18b), it is very much essential to know the corresponding accurate difference solution of u, which can be obtained using the formula (12).

4. ADI scheme and stability consideration

Now we consider the linear parabolic equation

$$v\left(u_{rr} + \frac{\alpha}{r}u_r + u_{zz}\right) = u_t + g(r,z,t), \quad 0 < r < 1, \quad t > 0, \tag{33}$$

where $v > 0$ represents diffusivity. For $\alpha = 1$ and 2, the equation above represents two-space dimensional diffusion equation in cylindrical and spherical symmetry, respectively.

Replacing the variables (x,y) by (r,z) and applying formula (12) to the differential equation (33), we obtain a linear difference scheme

$$\begin{bmatrix} 1 + \frac{1}{12}(1-6v\lambda+\lambda P_1)\,\delta_r^2 + \frac{1}{12}(1-6v\lambda)\delta_z^2 + \frac{1}{12}\left(\frac{\alpha h}{2r_l} + \lambda P_2\right)(2\mu_r\delta_r) \\ -\frac{\alpha v\lambda h}{24r_l}(\delta_z^2 2\mu_r\delta_r) - \frac{\lambda v}{12}\delta_r^2\delta_z^2 \end{bmatrix} u_{l,m}^{j+1},$$

$$= \begin{bmatrix} 1 + \frac{1}{12}(1+6v\lambda-\lambda P_1)\delta_r^2 + \frac{1}{12}(1+6v\lambda)\,\delta_z^2 + \frac{1}{12}\left(\frac{\alpha h}{2r_l} - \lambda P_2\right)(2\mu_r\delta_r) \\ +\frac{\alpha v\lambda h}{24r_l}(\delta_z^2 2\mu_r\delta_r) + \frac{\lambda v}{12}\delta_r^2\delta_z^2 \end{bmatrix} u_{l,m}^{j} - \frac{k}{12}\sum g, \tag{34}$$

where

$$\overline{g}_{l,m}^{j} = g(r_l, z_m, \overline{t}_j), \quad \overline{g}_{l\pm1/2,m}^{j} = g(r_{l\pm1/2}, z_m, \overline{t}_j) \quad \text{and} \quad \mu_r U_l = \tfrac{1}{2}(U_{l+1/2} + U_{l-1/2})$$

is the average difference operator with respect to r-direction, etc. and

$$P_1 = -\alpha v h \left(\frac{1}{r_{l+1/2}} - \frac{1}{r_{l-1/2}}\right) - \frac{v\alpha^2 h^2}{2r_l^2},$$

$$P_2 = -\alpha v h \left(\frac{1}{r_{l+1/2}} + \frac{1}{r_l} + \frac{1}{r_{l-1/2}}\right) - \frac{v\alpha^2 h^2}{4r_l}\left(\frac{1}{r_{l+1/2}} - \frac{1}{r_{l-1/2}}\right),$$

$$\sum g = 4\left(\overline{g}_{l+1/2,m}^{j} + \overline{g}_{l-1/2,m}^{j} + \overline{g}_{l,m+1/2}^{j} + \overline{g}_{l,m-1/2}^{j} - \overline{g}_{l,m}^{j}\right) + \frac{\alpha h}{r_l}\left(\overline{g}_{l+1/2,m}^{j} - \overline{g}_{l-1/2,m}^{j}\right).$$

The linear difference equation (34) requires solution of a system of equations with a large band width at each time level. It is also difficult to study the stability of such an equation.

We can rewrite Eq. (34) in product form as

$$\left[1 + \frac{1}{12}(1 - 6v\lambda + \lambda P_1)\delta_r^2 + \frac{1}{12}\left(\frac{\alpha h}{2r_l} + \lambda P_2\right)(2\mu_r\delta_r)\right]\left[1 + \frac{1}{12}(1 - 6v\lambda)\delta_z^2\right]u_{l,m}^{j+1}$$
$$= \left[1 + \frac{1}{12}(1 + 6v\lambda - \lambda P_1)\delta_r^2 + \frac{1}{12}\left(\frac{\alpha h}{2r_l} - \lambda P_2\right)(2\mu_r\delta_r)\right]\left[1 + \frac{1}{12}(1 + 6v\lambda)\delta_z^2\right]u_{l,m}^{j}$$
$$- \frac{k}{12}\sum g \equiv R_u. \tag{35}$$

The additional terms are of high orders and do not affect the accuracy of the scheme. In order to facilitate the computation, we may write (35) in two-step ADI method (see [13,7,2]) as

$$[1 + \tfrac{1}{12}(1 - 6v\lambda)\delta_z^2]u_{l,m}^{*} = R_u, \tag{36a}$$

$$\left[1 + \frac{1}{12}(1 - 6v\lambda + \lambda P_1)\delta_r^2 + \frac{1}{12}\left(\frac{\alpha h}{2r_l} + \lambda P_2\right)(2\mu_r\delta_r)\right]u_{l,m}^{j+1} = u_{l,m}^{*}, \tag{36b}$$

where $u_{l,m}^{*}$ is any intermediate value, and the intermediate boundary conditions required for the solution of $u_{l,m}^{*}$ can be obtained from (36b). The left-hand side matrices represented by (36a) and (36b) are tri-diagonal, hence, very easily solved in the solution region $0 < r < 1$, $t > 0$.

To study the stability of the difference scheme (35), we use the Von Neumann linear stability analysis. If we let $u_{l,m}^{j} = \xi^j e^{i\beta l} e^{i\gamma m}$ to be the value of $u_{l,m}^{j}$ at the node (l, m, j), where $i = \sqrt{-1}$, ξ is the amplitude and may be complex and β, γ are phase angles. The amplitude factor ξ, for stability, has to satisfy the inequality $|\xi| \leqslant 1$ for all β and γ in $[-\pi, \pi]$.

Substituting the expressions of $u_{l,m}^{j}$ and $u_{l,m}^{j+1}$ in the homogeneous part of Eq. (35), the amplification factor is found to be

$$\xi = \frac{A_1 A_2 + iA_3}{B_1 B_2 + iB_3}, \tag{37}$$

where

$$A_1 = 1 - \frac{1}{3}(1 + 6v\lambda - \lambda P_1)\sin^2\frac{\beta}{2},$$

$$A_2 = 1 - \frac{1}{3}(1 + 6v\lambda)\sin^2\frac{\gamma}{2},$$

$$A_3 = \frac{1}{6}\left(\frac{\alpha h}{2r_l} - \lambda P_2\right)\left[1 - \frac{1}{3}(1 + 6v\lambda)\sin^2\frac{\gamma}{2}\right]\sin\beta,$$

$$B_1 = 1 - \frac{1}{3}(1 - 6\nu\lambda + \lambda P_1)\sin^2\frac{\beta}{2},$$

$$B_2 = 1 - \frac{1}{3}(1 - 6\nu\lambda)\sin^2\frac{\gamma}{2},$$

$$B_3 = \frac{1}{6}\left(\frac{\alpha h}{2r_l} + \lambda P_2\right)\left[1 - \frac{1}{3}(1 - 6\nu\lambda)\sin^2\frac{\gamma}{2}\right]\sin\beta.$$

For stability it is required that $|\xi|^2 \leqslant 1$. Since $\max(\sin^2\beta/2) = \max(\sin^2\gamma/2) = 1$ and imposing this condition directly on (37), we found that the inequality $|\xi|^2 \leqslant 1$ is satisfied for all phase angles β and $\gamma \in [-\pi, \pi]$. Thus the ADI method (36a)–(36b) is unconditionally stable.

5. Experimental results

If we use the approximations (4), (5a), (6a), (7a), (8a) with $\theta = \frac{1}{2}$ into the differential equation (1), we get the central difference scheme

$$\varepsilon(\delta_x^2 + \delta_y^2)\overline{U}_{l,m}^j = h^2 f\left(x_l, y_m, \bar{t}_j, \overline{U}_{l,m}^j, \overline{U}_{x_{l,m}}^j, \overline{U}_{y_{l,m}}^j, \overline{U}_{t_{l,m}}^j\right) + \tilde{T}_{l,m}^j, \tag{38}$$

where $\tilde{T}_{l,m}^j = O(k^2h^2 + h^4)$.

In order to test the viability of the proposed methods, we have solved the following three problems, whose exact solutions are known. The initial and boundary conditions may be obtained using the exact solution as a test procedure. The linear equation has been solved using the ADI method, whereas the non-linear equations have been solved using the generalized Newton–Raphson method (see [3]). We have also compared the proposed method, with the corresponding central difference method (38) of $O(k^2 + h^2)$. All computations were carried out using double length arithmetic.

Example 1. The problem is to solve (33) with the exact solution $u(r, z, t) = e^{-\nu t}\cosh r.\cosh z$. The root mean square (RMS) errors for u, u_r and u_z are tabulated in Table 1 at $t = 1.0$ for $\nu = 0.01$, 0.001 and $\alpha = 1, 2$ for a fixed mesh ratio parameter $\lambda = 3.2$.

Example 2.

$$\nu(u_{xx} + u_{yy}) = u_t + u(u_x + u_y) \quad \text{(Burgers' Equation)} \quad 0 < x, \quad y < 1, \quad t > 0. \tag{39}$$

Table 1
Example 1: The RMS errors

h		$O(k^2+kh^2+h^4)$-ADI method				$O(k^2+h^2)$-ADI method			
		$\alpha=1$		$\alpha=2$		$\alpha=1$		$\alpha=2$	
		$\nu=0.01$	$\nu=0.001$	$\nu=0.01$	$\nu=0.001$	$\nu=0.01$	$\nu=0.001$	$\nu=0.01$	$\nu=0.001$
$\frac{1}{8}$	u	0.6276(−06)	0.8249(−07)	0.5483(−06)	0.6257(−07)	0.5852(−04)	0.6665(−05)	0.8762(−04)	0.9842(−05)
	u_r	0.9940(−05)	0.1045(−04)	0.1040(−04)	0.1079(−04)	0.7138(−04)	0.3177(−04)	0.1313(−03)	0.5119(−04)
	u_z	0.1736(−04)	0.1764(−04)	0.2491(−04)	0.2527(−04)	0.9402(−04)	0.3352(−04)	0.1419(−03)	0.4886(−04)
$\frac{1}{16}$	u	0.5041(−07)	0.6219(−08)	0.3265(−07)	0.3793(−08)	0.1414(−04)	0.1660(−05)	0.2128(−04)	0.2454(−05)
	u_r	0.6365(−06)	0.6634(−06)	0.6600(−06)	0.6887(−06)	0.3200(−04)	0.5656(−05)	0.4615(−04)	0.8084(−05)
	u_z	0.1099(−05)	0.1126(−05)	0.1583(−05)	0.1615(−05)	0.3957(−04)	0.5960(−05)	0.5965(−04)	0.6890(−05)
$\frac{1}{32}$	u	0.2828(−08)	0.3804(−09)	0.1918(−08)	0.2426(−09)	0.6116(−05)	0.7218(−06)	0.5618(−05)	0.6196(−06)
	u_r	0.2466(−07)	0.4216(−07)	0.4818(−07)	0.4992(−07)	0.1118(−04)	0.6814(−06)	0.1014(−04)	0.3480(−05)
	u_z	0.8215(−07)	0.8821(−07)	0.9286(−07)	0.1002(−06)	0.1414(−04)	0.7018(−06)	0.1818(−04)	0.2522(−05)

Table 2
Example 2: The RMS errors

h		$O(k^2 + kh^2 + h^4)$-method			$O(k^2 + h^2)$-method		
		$R_e = 10$	$R_e = 10^2$	$R_e = 10^3$	$R_e = 10$	$R_e = 10^2$	$R_e = 10^3$
$\frac{1}{8}$	u	0.2151(−02)	0.3002(−03)	0.4072(−05)	0.1001(−02)	0.2815(−03)	0.6055(−05)
	u_x	0.3923(−02)	0.1810(−02)	0.2545(−03)	0.3652(−02)	0.3640(−02)	0.7803(−03)
	u_y	0.3923(−02)	0.1810(−02)	0.2545(−03)	0.3652(−02)	0.3640(−02)	0.7803(−03)
$\frac{1}{16}$	u	0.1180(−03)	0.1754(−04)	0.2442(−06)	0.2452(−03)	0.6563(−04)	0.1526(−05)
	u_x	0.1066(−03)	0.8167(−04)	0.1393(−04)	0.1001(−02)	0.8886(−03)	0.1943(−03)
	u_y	0.1066(−03)	0.8167(−04)	0.1393(−04)	0.1001(−02)	0.8886(−03)	0.1943(−03)

Table 3
Example 3: The RMS errors

h		$O(k^2 + kh^2 + h^4)$-method			$O(k^2 + h^2)$-method	
		$\alpha = 10$	$\alpha = 50$	$\alpha = 100$	$\alpha = 10$	$\alpha = 50$ and 100
$\frac{1}{8}$	u	0.8054(−04)	0.3819(−03)	0.1136(−02)	0.2840(−02)	
	u_x	0.7753(−03)	0.3017(−02)	0.6479(−02)	0.7067(−02)	Over flow
	u_y	0.7753(−03)	0.3017(−02)	0.6479(−02)	0.7067(−02)	
$\frac{1}{16}$	u	0.4719(−05)	0.2345(−04)	0.7166(−04)	0.6611(−03)	
	u_x	0.4793(−04)	0.2034(−03)	0.4858(−03)	0.1768(−02)	Over flow
	u_y	0.4793(−04)	0.2034(−03)	0.4858(−03)	0.1768(−02)	

Table 4
Rate of convergence

Example 1: $h_1 = \frac{1}{16}$, $h_2 = \frac{1}{32}$	$\alpha = 1$		$\alpha = 2$	
	$\nu = 0.01$	$\nu = 0.001$	$\nu = 0.01$	$\nu = 0.001$
Convergence rate	4.15	4.03	4.08	3.96
Example 2: $h_1 = \frac{1}{8}, h_2 = \frac{1}{16}$	$R_e = 10$	$R_e = 10^2$	$R_e = 10^3$	
Convergence rate	4.18	4.09	4.05	
Example 3: $h_1 = \frac{1}{8}, h_2 = \frac{1}{16}$	$\alpha = 10$	$\alpha = 50$	$\alpha = 100$	
Convergence rate	4.09	4.02	3.98	

The exact solution is given by

$$u(x, y, t) = \frac{2\nu\pi \sin(\pi(x + y))e^{-2\nu\pi^2 t}}{2 + \cos(\pi(x + y))e^{-2\nu\pi^2 t}},$$

where $\nu = (1/R_e) > 0$. The RMS errors for u, u_x and u_y are tabulated in Table 2 at $t = 1.0$ for various values of R_e for a fixed mesh ratio parameter $\lambda = 1.6$.

Example 3.

$$u_{xx} + u_{yy} = u_t + \alpha u(u_x + u_y) + f(x, y, t), \quad 0 < x, \quad y < 1, \quad t > 0. \tag{40}$$

The exact solution is given by $u(x, y, t) = e^{-t} \sin(\pi x) \sin(\pi y)$. The RMS errors for u, u_x and u_y are tabulated in Table 3 at $t = 1.0$ for various values of α for a fixed mesh ratio parameter $\lambda = 1.6$.

The rates of convergence of the proposed difference scheme (12) for u in each case are computed in Table 4 using the formula

$$\frac{\log(e_{h_1}/e_{h_2})}{\log(h_1/h_2)}, \tag{41}$$

where e_{h_1} and e_{h_2} are the RMS errors for u for h_1 and h_2, respectively.

6. Concluding remarks

In this article, we have developed a new two-level 9-point implicit finite difference method of $O(k^2 + kh^2 + h^4)$ based on arithmetic average discretization for the solution of 2-D non-linear parabolic partial differential equation and the estimates of first-order derivatives $(\partial u/\partial x)$ and $(\partial u/\partial y)$. Although the proposed methods involve more algebra, the methods are directly applicable to singular problems without any modification in the original scheme, which is an added advantage. The ADI scheme (36) for the linear equation has been proved to be unconditionally stable with respect to initial values. To the authors knowledge no stability theory for the non-linear difference scheme has been discussed in the literature so far. For computation, RMS errors for u have been calculated at $t = 1.0$. For a given value of λ and h, the value of k can be computed from $k = \lambda h^2$. For example, for $h = \frac{1}{8}$, $\lambda = 1.6$ and 3.2, the value of $k = \frac{1}{40}$ and $\frac{1}{20}$, respectively, that is, for $\lambda = 1.6$ and 3.2 we require 40 and 20 time steps to get the same numerical value of u of required accuracy at $t = 1.0$. Use of larger value of λ indicates that we can use the large time step and less computer time to achieve the required accuracy. In most of the cases, non-linear difference schemes are stable for $0 < \lambda < 1$. But fortunately, in our case, non-linear difference schemes are stable for $\lambda = 1.6 > 1$. Results from our numerical experiments indicate that the proposed high order methods are computationally more efficient than the corresponding difference methods of $O(k^2 + h^2)$. The numerical results confirm that the proposed methods produce oscillation free solutions for $0 < \varepsilon \ll 1$ and the rate of convergence is indeed nearly equal to 4.0.

Acknowledgment

The authors thank the reviewers for their valuable suggestions, which greatly improved the quality of the paper.

References

[1] M.M. Chawla, P.N. Shivakumar, An efficient finite difference method for two-point boundary value problems, Neural Parallel Sci. Comput. 4 (1996) 387–396.
[2] E.G. D'yakonov, Difference schemes of second-order accuracy with a splitting operator for parabolic equations without mixed derivatives, Zh. Vychisl. Mat. I Mat. Fiz. 4 (1964) 935–941.
[3] L.A. Hageman, D.M. Young, Applied Iterative Methods, Academic Press, New York, 1981.
[4] S.R.K. Iyenger, R.P. Manohar, High order difference methods for heat equation in polar cylindrical coordinates, J. Comput. Phys. 77 (1988) 425–438.
[5] J.C. Kalita, D.C. Dalal, A.K. Dass, A class of higher order compact schemes for the unsteady two-dimensional convection–diffusion equation with variable convection coefficients, Internat. J. Numer. Methods Fluids 38 (2002) 1111–1131.
[6] S. Karaa, J. Zhang, High order ADI method for solving unsteady convection–diffusion problems, J. Comput. Phys. 198 (2004) 1–9.
[7] M. Lees, Alternating direction and semi-explicit difference methods for parabolic partial differential equations, Numer. Math. 3 (1961) 398–405.
[8] R.K. Mohanty, M.K. Jain, High accuracy difference scheme for the system of two space non-linear parabolic differential equations with mixed derivatives and variable coefficients, J. Comput. Appl. Math. 70 (1996) 15–32.
[9] R.K. Mohanty, M.K. Jain, D. Kumar, Single cell discretization of $O(kh^2 + h^4)$ for the estimates of $(\partial u/\partial n)$ for two space dimensional quasi-linear parabolic equations, Numer. Methods Partial Differential Equations 17 (2001) 250–261.
[10] R.K. Mohanty, S. Singh, Non-uniform mesh arithmetic average discretization for parabolic initial boundary value problems, Neural Parallel Sci. Comput. 13 (2005) 401–416.
[11] R.K. Mohanty, S. Singh, A new fourth order discretization for singularly perturbed two-dimensional non-linear elliptic boundary value problems, Appl. Math. Comput. 175 (2006) 1400–1414.
[12] B.J. Noye, H.H. Tan, Finite difference method for solving the two-dimensional advection–diffusion equation, Internat. J. Numer. Methods Fluids 26 (1988) 1615–1629.
[13] D.W. Peaceman, H.H. Rachfond Jr., The numerical solution of parabolic and elliptic differential equations, J. Soc. Ind. Appl. Math. 3 (1959) 28–41.

[14] P.J. Roach, Computational Fluid Dynamics, Hermosa, Albuquerque, NM, 1976.
[15] W.F. Spotz, G.F. Carey, Extension of high-order compact schemes to time-dependent problems, Numer. Methods Partial Differential Equations 17 (2001) 657–672.
[16] J.W. Stephenson, Single cell discretizations of order two and four for biharmonic problems, J. Comput. Phys. 55 (1984) 65–80.
[17] J. Zhang, J.J. Zhao, Unconditionally stable finite difference scheme and iterative solution of 2-D microscale heat transport equation, J. Comput. Phys. 170 (2001) 261–275.